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Some Grüss-type results via Pompeiu's-like inequalities

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Abstract In this paper, some Grüss-type results via Pompeiu's-like inequalities are proved.

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الملخص

في هذه الورقة، تم إثبات بعض النتائج من نوع جْرُسْ باستخدام متباينات من عائلة متباينة بومبوي.

1 Introduction

In 1946, Pompeiu [18] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [18, p.83]).

Theorem 1.1 (Pompeiu [18]) *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

The following inequality is useful to derive some Ostrowski-type inequalities; see [9].

Corollary 1.2 (Pompeiu's inequality) *With the assumptions of Theorem 1.1 and if $\|f - \ell f'\|_\infty = \sup_{t\in(a,b)} |f(t) - tf'(t)| < \infty$ where $\ell(t) = t$, $t \in [a, b]$, then*

$$|tf(x) - xf(t)| \leq \|f - \ell f'\|_\infty |x - t| \tag{1.2}$$

for any $t, x \in [a, b]$.

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The inequality (1.2) was obtained by the author in [9].

For other Ostrowski-type inequalities concerning the p-norms $\left\| f - \ell f' \right\|_p$, see [1,2,17,19].

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) \mathrm{d}t - \frac{1}{(b-a)^2} \int_a^b f(t) \mathrm{d}t \int_a^b g(t) \mathrm{d}t. \tag{1.3}$$

Grüss [10] showed that

$$|C(f, g)| \le \frac{1}{4} (M - m)(N - n), \tag{1.4}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a, b]. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in (1.3) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known, result, though it was obtained by Čebyšev [7], states that

$$|C(f, g)| \le \frac{1}{12} \left\| f' \right\|_\infty \left\| g' \right\|_\infty (b-a)^2, \tag{1.6}$$

provided that f', g' exist and are continuous on $[a, b]$ and $\left\| f' \right\|_\infty = \sup_{t \in [a,b]} \left| f'(t) \right|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.6) also holds if $f, g : [a, b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty [a, b]$, while $\left\| f' \right\|_\infty = \operatorname{ess\,sup}_{t \in [a,b]} \left| f'(t) \right|$.

A mixture between Grüss' result (1.4) and Čebyšev's one (1.6) is the following inequality obtained by Ostrowski [15]:

$$|C(f, g)| \le \frac{1}{8} (b-a)(M-m) \left\| g' \right\|_\infty, \tag{1.7}$$

provided that f is *Lebesgue integrable* and satisfies (1.5), while g is absolutely continuous and $g' \in L_\infty [a, b]$. The constant $\frac{1}{8}$ is best possible in (1.7).

The case of *Euclidean norms* of the derivative was considered by Lupaş [12], in which he proved that

$$|C(f, g)| \le \frac{1}{\pi^2} \left\| f' \right\|_2 \left\| g' \right\|_2 (b-a), \tag{1.8}$$

provided that f, g are absolutely continuous and $f', g' \in L_2 [a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, Cerone and Dragomir [3] have proved the following results:

$$|C(f, g)| \le \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \right|^p \mathrm{d}t \right)^{\frac{1}{p}}, \tag{1.9}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f, g)| \le \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_1 \cdot \frac{1}{b-a} \operatorname*{ess\,sup}_{t \in [a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \right|, \tag{1.10}$$

provided that $f \in L_p [a, b]$ and $g \in L_q [a, b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$).

Notice that for $q = \infty$, $p = 1$ in (1.9), we obtain

$$\begin{aligned} |C(f, g)| &\le \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \right| \mathrm{d}t \\ &\le \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) \mathrm{d}s \right| \mathrm{d}t \end{aligned} \tag{1.11}$$

and, if g satisfies (1.5), then

$$\begin{aligned}|C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|\mathrm{d}t \\ &\leq \left\|g-\frac{n+N}{2}\right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|\mathrm{d}t \\ &\leq \frac{1}{2}(N-n)\cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|\mathrm{d}t. \end{aligned} \tag{1.12}$$

The inequality between the first and the last term in (1.12) has been obtained by Cheng and Sun [8]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [5,6,11,13,14,16,20] and the references therein.

In this paper, some Grüss-type results via Pompeiu's-like inequalities are proved.

2 Some Pompeiu's-type inequalities

We can generalize the above inequality for the larger class of functions that are absolutely continuous and complex valued as well as for other norms of the difference $f-\ell f'$.

Theorem 2.1 *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b>a>0$. Then for any $t,x\in[a,b]$, we have*

$$|tf(x)-xf(t)| \leq \begin{cases} \left\|f-\ell f'\right\|_\infty |x-t| & \text{if } f-\ell f'\in L_\infty[a,b], \\ \left(\frac{1}{2q-1}\right)^{1/q}\left\|f-\ell f'\right\|_p \left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q} & \text{if } f-\ell f'\in L_p[a,b] \\ & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \left\|f-\ell f'\right\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, & \end{cases} \tag{2.1}$$

or equivalently

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right| \leq \begin{cases} \left\|f-\ell f'\right\|_\infty \left|\frac{1}{t}-\frac{1}{x}\right| & \text{if } f-\ell f'\in L_\infty[a,b], \\ \left(\frac{1}{2q-1}\right)^{1/q}\left\|f-\ell f'\right\|_p \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q} & \text{if } f-\ell f'\in L_p[a,b] \\ & p>1, \\ & \frac{1}{p}+\frac{1}{q}=1, \\ \left\|f-\ell f'\right\|_1 \frac{1}{\min\{t^2,x^2\}}. & \end{cases} \tag{2.2}$$

Proof If f is absolutely continuous, then f/ℓ is absolutely continuous on the interval $[a,b]$ that does not contain 0 and

$$\int_t^x \left(\frac{f(s)}{s}\right)'\mathrm{d}s = \frac{f(x)}{x}-\frac{f(t)}{t}$$

for any $t,x\in[a,b]$ with $x\neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{s}\right)'\mathrm{d}s = \int_t^x \frac{f'(s)s-f(s)}{s^2}\mathrm{d}s,$$

we get the following identity:

$$tf(x)-xf(t) = xt\int_t^x \frac{f'(s)s-f(s)}{s^2}\mathrm{d}s \tag{2.3}$$

for any $t, x \in [a, b]$.

We notice that the equality (2.3) was proved for the smaller class of differentiable function and in a different manner in [17].

Taking the modulus in (2.3), we have

$$
\begin{aligned}
|tf(x) - xf(t)| &= \left| xt \int_t^x \frac{f'(s)s - f(s)}{s^2} \mathrm{d}s \right| \\
&\leq xt \left| \int_t^x \left| \frac{f'(s)s - f(s)}{s^2} \right| \mathrm{d}s \right| := I,
\end{aligned} \tag{2.4}
$$

and utilizing Hölder's integral inequality we deduce

$$
\begin{aligned}
I &\leq xt \begin{cases} \sup_{s\in[t,x]([x,t])} \left| f'(s)s - f(s) \right| \left| \int_t^x \frac{1}{s^2} \mathrm{d}s \right|, \\ \left| \int_t^x \left| f'(s)s - f(s) \right|^p \mathrm{d}s \right|^{1/p} \left| \int_t^x \frac{1}{s^{2q}} \mathrm{d}s \right|^{1/q} & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \left| \int_t^x \left| f'(s)s - f(s) \right| \mathrm{d}s \right| \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{s^2} \right\}, \end{cases} \\
&= \begin{cases} \left\| f - \ell f' \right\|_\infty |x - t|, \\ \left(\frac{1}{2q-1} \right)^{1/q} \left\| f - \ell f' \right\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \left\| f - \ell f' \right\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases}
\end{aligned} \tag{2.5}
$$

and the inequality (2.2) is proved. □

Remark 2.2 The first inequality in (2.1) also holds in the same form for $0 > b > a$.

3 Some Grüss-type inequalities

We have the following result of Grüss type.

Theorem 3.1 *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_\infty[a, b]$, then*

$$
\begin{aligned}
&\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \mathrm{d}t - \int_a^b t f(t) \mathrm{d}t \int_a^b t g(t) \mathrm{d}t \right| \\
&\quad \leq \frac{1}{12} (b-a)^4 \left\| f - \ell f' \right\|_\infty \left\| g - \ell g' \right\|_\infty .
\end{aligned} \tag{3.1}
$$

The constant $\frac{1}{12}$ is best possible.

Proof From the first inequality in (2.1), we have

$$
\begin{aligned}
&\left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t)) \mathrm{d}t\mathrm{d}x \right| \\
&\quad \leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))| \mathrm{d}t\mathrm{d}x \\
&\quad \leq \left\| f - \ell f' \right\|_\infty \left\| g - \ell g' \right\|_\infty \int_a^b \int_a^b (x-t)^2 \mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{3.2}
$$

Observe that

$$
\begin{aligned}
&\int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t))\,\mathrm{d}t\mathrm{d}x \\
&= \int_a^b \int_a^b \left[t^2 f(x) g(x) + x^2 f(t) g(t) - tg(t) xf(x) - f(t) txg(x)\right] \mathrm{d}t\mathrm{d}x \\
&= 2\left[\int_a^b t^2 \mathrm{d}t \int_a^b f(t) g(t)\,\mathrm{d}t - \int_a^b tf(t)\,\mathrm{d}t \int_a^b tg(t)\,\mathrm{d}t\right]
\end{aligned}
$$

and

$$
\int_a^b \int_a^b (x-t)^2 \,\mathrm{d}t\mathrm{d}x = \frac{1}{3}\int_a^b \left[(b-x)^3 + (x-a)^3\right] \mathrm{d}x = \frac{1}{6}(b-a)^4 .
$$

Utilizing the inequality (3.2), we deduce the desired result (3.1).

Now, assume that the inequality (3.1) holds with a constant $B > 0$ instead of $\frac{1}{12}$, i.e.,

$$
\begin{aligned}
&\left|\frac{b^3-a^3}{3}\int_a^b f(t) g(t)\,\mathrm{d}t - \int_a^b tf(t)\,\mathrm{d}t \int_a^b tg(t)\,\mathrm{d}t\right| \\
&\leq B(b-a)^4 \left\|f - \ell f'\right\|_\infty \left\|g - \ell g'\right\|_\infty .
\end{aligned} \tag{3.3}
$$

If we take $f(t) = g(t) = 1$, $t \in [a, b]$, then

$$
\begin{aligned}
&\frac{b^3-a^3}{3}\int_a^b f(t) g(t)\,\mathrm{d}t - \int_a^b tf(t)\,\mathrm{d}t \int_a^b tg(t)\,\mathrm{d}t \\
&= \frac{b^3-a^3}{3}(b-a) - \left(\frac{b^2-a^2}{2}\right)^2 = \frac{1}{12}(b-a)^4
\end{aligned}
$$

and

$$
\left\|f - \ell f'\right\|_\infty = \left\|g - \ell g'\right\|_\infty = 1
$$

and by (3.3) we get $B \geq \frac{1}{12}$, which proves the sharpness of the constant. □

The following result for the complementary (p, q)-norms, with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, holds.

Theorem 3.2 *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f' \in L_p[a, b]$, $g' \in L_q[a, b]$ with $p, q > 1$, $p, q \neq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$
\begin{aligned}
&\left|\frac{b^3-a^3}{3}\int_a^b f(t) g(t)\,\mathrm{d}t - \int_a^b tf(t)\,\mathrm{d}t \int_a^b tg(t)\,\mathrm{d}t\right| \\
&\leq \frac{1}{2(2q-1)^{1/q}(2p-1)^{1/p}} \left\|f - \ell f'\right\|_p \left\|g - \ell g'\right\|_q M_q^{1/q}(a, b)\, M_p^{1/p}(a, b),
\end{aligned} \tag{3.4}
$$

where

$$
M_q(a, b) := \int_a^b \int_a^b \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right| \mathrm{d}t\mathrm{d}x .
$$

We have the bounds

$$
M_q(a, b) \leq (b-a)\, N_q^{1/2}(a, b)
$$

and

$$
M_p(a, b) \leq (b-a)\, N_p^{1/2}(a, b)
$$

where, for $r > 1$,

$$N_r(a,b) := \begin{cases} 2\left(\frac{b^{2r+1}-a^{2r+1}}{2r+1} \cdot \frac{b^{-2r+3}-a^{-2r+3}}{-2r+3} - \left(\frac{b^2-a^2}{2}\right)^2\right), r \neq \frac{3}{2} \\ \left(b^2-a^2\right)\left(\frac{b^2+a^2}{2} \cdot \ln \frac{b}{a} - \frac{b^2-a^2}{2}\right), r = \frac{3}{2}. \end{cases}$$

Proof From the second inequality in (2.1), we have

$$|tf(x) - xf(t)| \leq \frac{1}{(2q-1)^{1/q}} \left\| f - \ell f' \right\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q}$$

and

$$|tg(x) - xg(t)| \leq \frac{1}{(2p-1)^{1/p}} \left\| g - \ell g' \right\|_q \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p}$$

for any $t, x \in [a,b]$.

If we multiply these inequalities and integrate, then we get

$$\begin{aligned} &\left| \int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t))\, \mathrm{d}t\mathrm{d}x \right| \\ &\leq \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))|\, \mathrm{d}t\mathrm{d}x \\ &\leq \frac{1}{(2q-1)^{1/q}(2p-1)^{1/p}} \left\| f - \ell f' \right\|_p \left\| g - \ell g' \right\|_q \\ &\quad \times \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} \mathrm{d}t\mathrm{d}x. \end{aligned} \tag{3.5}$$

Utilizing Hölder's integral inequality for double integrals, we have

$$\begin{aligned} &\int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} \mathrm{d}t\mathrm{d}x \\ &\leq \left(\int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| \mathrm{d}t\mathrm{d}x \right)^{1/q} \left(\int_a^b \int_a^b \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right| \mathrm{d}t\mathrm{d}x \right)^{1/p} \\ &= M_q^{1/q}(a,b)\, M_p^{1/p}(a,b) \end{aligned} \tag{3.6}$$

for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Utilizing Cauchy–Bunyakowsky–Schwarz integral inequality for double integrals, we have

$$\begin{aligned} M_q(a,b) &= \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| \mathrm{d}t\mathrm{d}x \\ &\leq \left(\int_a^b \int_a^b \mathrm{d}t\mathrm{d}x \right)^{1/2} \left(\int_a^b \int_a^b \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 \mathrm{d}t\mathrm{d}x \right)^{1/2} \\ &= (b-a) \left(\int_a^b \int_a^b \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 \mathrm{d}t\mathrm{d}x \right)^{1/2}. \end{aligned}$$

Observe that

$$
\begin{aligned}
N_q(a,b) &:= \int_a^b \int_a^b \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 \mathrm{d}t\mathrm{d}x \\
&= \int_a^b \int_a^b \frac{x^{2q}}{t^{2(q-1)}} \mathrm{d}t\mathrm{d}x - 2\int_a^b \int_a^b \frac{x^q}{t^{q-1}} \frac{t^q}{x^{q-1}} \mathrm{d}t\mathrm{d}x + \int_a^b \int_a^b \frac{t^{2q}}{x^{2(q-1)}} \mathrm{d}t\mathrm{d}x \\
&= 2\int_a^b x^{2q}\mathrm{d}x \int_a^b t^{-2(q-1)}\mathrm{d}t - 2\left(\int_a^b x\mathrm{d}x \right)^2 \\
&= 2\left(\frac{b^{2q+1} - a^{2q+1}}{2q+1} \cdot \frac{b^{-2q+3} - a^{-2q+3}}{-2q+3} - \left(\frac{b^2 - a^2}{2} \right)^2 \right),
\end{aligned}
$$

provided $q \neq \frac{3}{2}$.

If $q = \frac{3}{2}$, then

$$
N_q(a,b) = \left(b^2 - a^2\right) \left[\frac{b^2 + a^2}{2} \cdot \ln \frac{b}{a} - \frac{b^2 - a^2}{2} \right].
$$

Therefore,

$$
M_q(a,b) \leq (b-a) N_q^{1/2}(a,b)
$$

and, similarly,

$$
M_p(a,b) \leq (b-a) N_p^{1/2}(a,b).
$$

□

Remark 3.3 The double integral

$$
M_q(a,b) := \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| \mathrm{d}t\mathrm{d}x
$$

can be computed exactly by iterating the integrals. However, the final form is too complicated to be stated here.

The Euclidian norms case is as follows:

Theorem 3.4 *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $b > a > 0$. If $f', g' \in L_2[a,b]$, then*

$$
\begin{aligned}
&\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \mathrm{d}t - \int_a^b t f(t) \mathrm{d}t \int_a^b t g(t) \mathrm{d}t \right| \\
&\quad \leq \frac{1}{9} \left\| f - \ell f' \right\|_2 \left\| g - \ell g' \right\|_2 \left[\left(b^3 + a^3\right) \ln \frac{b}{a} - \frac{2}{3}\left(b^3 - a^3\right) \right].
\end{aligned}
\tag{3.7}
$$

Proof From the second inequality in (2.1), we have

$$
|t f(x) - x f(t)| \leq \frac{1}{\sqrt{3}} \left\| f - \ell f' \right\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}
$$

and

$$
|t g(x) - x g(t)| \leq \frac{1}{\sqrt{3}} \left\| g - \ell g' \right\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}
$$

for any $t, x \in [a,b]$.

If we multiply these inequalities and integrate, then we get

$$\begin{aligned}
&\left|\int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t))\,\mathrm{d}t\mathrm{d}x\right| \\
&\quad \le \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))|\,\mathrm{d}t\mathrm{d}x \\
&\quad \le \frac{1}{3}\left\|f - \ell f'\right\|_2 \left\|g - \ell g'\right\|_2 \int_a^b \int_a^b \left|\frac{x^2}{t} - \frac{t^2}{x}\right| \mathrm{d}t\mathrm{d}x. \qquad (3.8)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_a^b \int_a^b \left|\frac{x^2}{t} - \frac{t^2}{x}\right| \mathrm{d}t\mathrm{d}x \\
&\quad = \int_a^b \left(\int_a^x \left(\frac{x^2}{t} - \frac{t^2}{x}\right)\mathrm{d}t + \int_x^b \left(\frac{t^2}{x} - \frac{x^2}{t}\right)\mathrm{d}t\right)\mathrm{d}x \\
&\quad = \int_a^b \left(x^2(2\ln x - \ln a - \ln b) + \frac{b^3 + a^3 - 2x^3}{3x}\right)\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b x^2(2\ln x - \ln a - \ln b)\,\mathrm{d}x \\
&\quad = \int_a^b 2x^2 \ln x\,\mathrm{d}x - \ln(ab)\int_a^b x^2\mathrm{d}x \\
&\quad = \frac{\left(b^3 + a^3\right)\ln\frac{b}{a}}{3} - \frac{2}{9}\left(b^3 - a^3\right),
\end{aligned}$$

while

$$\int_a^b \frac{b^3 + a^3 - 2x^3}{3x}\mathrm{d}x = \frac{\left(b^3 + a^3\right)\ln\frac{b}{a}}{3} - \frac{2}{9}\left(b^3 - a^3\right),$$

then we conclude that

$$\int_a^b \int_a^b \left|\frac{x^2}{t} - \frac{t^2}{x}\right| \mathrm{d}t\mathrm{d}x = \frac{2}{3}\left[\left(b^3 + a^3\right)\ln\frac{b}{a} - \frac{2}{3}\left(b^3 - a^3\right)\right].$$

Making use of the inequality (3.8), we deduce the desired result (3.7). □

Remark 3.5 It is an open question to the author if $\frac{1}{9}$ is best possible in (3.7).

Theorem 3.6 *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. Then,*

$$\begin{aligned}
&\left|\frac{b^3 - a^3}{3}\int_a^b f(t)\,g(t)\,\mathrm{d}t - \int_a^b tf(t)\,\mathrm{d}t \int_a^b tg(t)\,\mathrm{d}t\right| \\
&\quad \le \left\|f - \ell f'\right\|_1 \left\|g - \ell g'\right\|_1 \frac{2b^3 + a^3 - 3ab^2}{6a}. \qquad (3.9)
\end{aligned}$$

Proof From the third inequality in (2.1), we have

$$\begin{aligned}
&\left|\int_a^b \int_a^b (tf(x) - xf(t))(tg(x) - xg(t))\,\mathrm{d}t\mathrm{d}x\right| \\
&\quad \le \int_a^b \int_a^b |(tf(x) - xf(t))(tg(x) - xg(t))|\,\mathrm{d}t\mathrm{d}x \\
&\quad \le \left\|f - \ell f'\right\|_1 \left\|g - \ell g'\right\|_1 \int_a^b \int_a^b \left(\frac{\max\{t, x\}}{\min\{t, x\}}\right)^2 \mathrm{d}t\mathrm{d}x. \qquad (3.10)
\end{aligned}$$

Observe that

$$
\begin{aligned}
&\int_a^b \int_a^b \left(\frac{\max\{t,x\}}{\min\{t,x\}}\right)^2 \mathrm{d}t\mathrm{d}x \\
&= \int_a^b \left[\int_a^x \left(\frac{\max\{t,x\}}{\min\{t,x\}}\right)^2 \mathrm{d}t + \int_x^b \left(\frac{\max\{t,x\}}{\min\{t,x\}}\right)^2 \mathrm{d}t\right] \mathrm{d}x \\
&= \int_a^b \left[\int_a^x \left(\frac{x}{t}\right)^2 \mathrm{d}t + \int_x^b \left(\frac{t}{x}\right)^2 \mathrm{d}t\right] \mathrm{d}x \\
&= \frac{2b^3 + a^3 - 3ab^2}{6a},
\end{aligned}
$$

which together with (3.10) produces the desired inequality (3.9). □

4 Some related results

The following result holds.

Theorem 4.1 *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_\infty [a, b]$, then*

$$
\begin{aligned}
&\left|(b-a)\int_a^b \frac{f(t)\,g(t)}{t^2}\mathrm{d}t - \int_a^b \frac{f(t)}{t}\mathrm{d}t \int_a^b \frac{g(t)}{t}\mathrm{d}t\right| \\
&\le (b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)\,G^2(a,b)} \left\|f - \ell f'\right\|_\infty \left\|g - \ell g'\right\|_\infty,
\end{aligned} \tag{4.1}
$$

where $G(a, b) := \sqrt{ab}$ is the geometric mean and

$$L(a,b) := \frac{b-a}{\ln b - \ln a}$$

is the Logarithmic mean.

The inequality (4.1) is sharp.

Proof From the first inequality in (2.2), we have

$$
\begin{aligned}
&\left|\left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right)\right| \\
&\le \left\|f - \ell f'\right\|_\infty \left\|f - \ell f'\right\|_\infty \left(\frac{1}{t} - \frac{1}{x}\right)^2
\end{aligned} \tag{4.2}
$$

for any $t, x \in [a, b]$.

Integrating this inequality on $[a, b]^2$, we get

$$
\begin{aligned}
&\left|\int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right) \mathrm{d}t\mathrm{d}x\right| \\
&\int_a^b \int_a^b \left|\left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right)\right| \mathrm{d}t\mathrm{d}x \\
&\le \left\|f - \ell f'\right\|_\infty \left\|f - \ell f'\right\|_\infty \int_a^b \int_a^b \left(\frac{1}{t} - \frac{1}{x}\right)^2 \mathrm{d}t\mathrm{d}x.
\end{aligned} \tag{4.3}
$$

We have

$$\int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right) \mathrm{d}t\mathrm{d}x$$
$$= 2\left[(b-a)\int_a^b \frac{f(t)\,g(t)}{t^2}\mathrm{d}t - \int_a^b \frac{f(t)}{t}\mathrm{d}t \int_a^b \frac{g(t)}{t}\mathrm{d}t\right]$$

and

$$\int_a^b \int_a^b \left(\frac{1}{t} - \frac{1}{x}\right)^2 \mathrm{d}t\mathrm{d}x = 2(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)\,G^2(a,b)}.$$

Making use of (4.3), we get the desired result (4.1).

If we take $f(t) = g(t) = 1$, then we have

$$(b-a)\int_a^b \frac{f(t)\,g(t)}{t^2}\mathrm{d}t - \int_a^b \frac{f(t)}{t}\mathrm{d}t \int_a^b \frac{g(t)}{t}\mathrm{d}t$$
$$= (b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)\,G^2(a,b)}$$

and

$$\|f - \ell f'\|_\infty = \|g - \ell g'\|_\infty = 1,$$

and we obtain in both sides of (4.1) the same quantity

$$(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)\,G^2(a,b)}.$$

□

The case of Euclidian norms is as follows:

Theorem 4.2 *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $b > a > 0$. If $f', g' \in L_2[a,b]$, then*

$$\left|(b-a)\int_a^b \frac{f(t)\,g(t)}{t^2}\mathrm{d}t - \int_a^b \frac{f(t)}{t}\mathrm{d}t \int_a^b \frac{g(t)}{t}\mathrm{d}t\right|$$
$$\leq \frac{1}{6}\|f - \ell f'\|_2 \|g - \ell g'\|_2 \frac{(b-a)^3}{a^2 b^2}. \tag{4.4}$$

Proof From the second inequality in (2.2) for $p = q = 2$, we have

$$\left|\frac{f(x)}{x} - \frac{f(t)}{t}\right| \leq \frac{1}{\sqrt{3}}\|f - \ell f'\|_2 \left|\frac{1}{t^3} - \frac{1}{x^3}\right|^{1/2} \tag{4.5}$$

and

$$\left|\frac{g(x)}{x} - \frac{g(t)}{t}\right| \leq \frac{1}{\sqrt{3}}\|g - \ell g'\|_2 \left|\frac{1}{t^3} - \frac{1}{x^3}\right|^{1/2} \tag{4.6}$$

for any $t, x \in [a,b]$.

On multiplying (4.5) with (4.6), we derive

$$\left|\left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right)\right| \leq \frac{1}{3}\|f - \ell f'\|_2 \|g - \ell g'\|_2 \left|\frac{1}{t^3} - \frac{1}{x^3}\right| \tag{4.7}$$

for any $t, x \in [a,b]$.

Integrating this inequality on $[a, b]^2$, we get

$$\begin{aligned}&\left|\int_a^b\int_a^b\left(\frac{f(x)}{x}-\frac{f(t)}{t}\right)\left(\frac{g(x)}{x}-\frac{g(t)}{t}\right)\mathrm{d}t\mathrm{d}x\right|\\&\le\int_a^b\int_a^b\left|\left(\frac{f(x)}{x}-\frac{f(t)}{t}\right)\left(\frac{g(x)}{x}-\frac{g(t)}{t}\right)\right|\mathrm{d}t\mathrm{d}x\\&\le\frac{1}{3}\left\|f-\ell f'\right\|_2\left\|g-\ell g'\right\|_2\int_a^b\int_a^b\left|\frac{1}{t^3}-\frac{1}{x^3}\right|\mathrm{d}t\mathrm{d}x.\end{aligned}\tag{4.8}$$

We have

$$\begin{aligned}\int_a^b\int_a^b\left|\frac{1}{t^3}-\frac{1}{x^3}\right|\mathrm{d}t\mathrm{d}x&=\int_a^b\left[\int_a^x\left(\frac{1}{t^3}-\frac{1}{x^3}\right)\mathrm{d}t+\int_x^b\left(\frac{1}{x^3}-\frac{1}{t^3}\right)\mathrm{d}t\right]\mathrm{d}x\\&=\int_a^b\left[\int_a^x\left(\frac{1}{t^3}-\frac{1}{x^3}\right)\mathrm{d}t+\int_x^b\left(\frac{1}{x^3}-\frac{1}{t^3}\right)\mathrm{d}t\right]\mathrm{d}x=\frac{(b-a)^3}{a^2b^2}.\end{aligned}$$

From (4.8), we then obtain the desired result (4.4). □

Remark 4.3 It is an open question to the author if $\frac{1}{6}$ is the best possible constant in (4.4).

The interested reader may obtain other similar results in terms of the norms $\left\|f-\ell f'\right\|_p\left\|g-\ell g'\right\|_q$ with $p, q > 1$, $p, q \neq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. However, the details are omitted.

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